General Univalence Criteria in the Disk: Extensions and Extremal Function*

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Abstract

Many classical univalence criteria depending on the Schwarzian derivative are special cases of a result, proved in [18], involving both conformal mappings and conformal metrics. The classical theorems for analytic function on the disk emerge by choosing appropriate conformal metrics and computing a generalized Schwarzian. The results in this paper address questions of extending functions which satisfy the general univalence criterion; continuous extensions to the closure of the disk, and homeomorphic and quasiconformal extensions to the sphere. The main tool is the convexity of an associated function along geodesics of the metric. The other important aspect of this study is an extremal function associated with a given criterion, along with its associated extremal geodesics. An extremal function for a criterion is one whose image is not a Jordan domain. An extremal geodesic joins points on the boundary which map to the same point in the image. We show that, for the general criterion, the image of an extremal geodesic under an extremal function is a euclidean circle.

1 Introduction

In this paper we study some geometric aspects of univalence criteria depending on the Schwarzian derivative in a fairly general setting. The Schwarzian derivative of an analytic function f is defined by

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

Let $\mathbf{D}$ denote the unit disk in the complex plane. We consider analytic or meromorphic functions defined on $\mathbf{D}$ and metrics on $\mathbf{D}$ of nonpositive curvature that are conformal to the euclidean metric. Our main concerns are with extending maps satisfying the general univalence criterion in Theorem 1, below, to $\overline{\mathbf{D}}$ and to $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, and also with geometric properties of extremal functions for the criterion. We make systematic use of convexity coming from comparison theorems for differential equations and inequalities.

Univalence Criteria In the paper [17] that started the whole subject, Nehari proved that either of the conditions

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2} \tag{1.1}$$

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or

$$|Sf(z)| \leq \frac{\pi^2}{2}, \tag{1.2}$$

is sufficient for f to be univalent in $\mathbf{D}$. The constants $\pi^2/2$ in (1.2) and 2 in the numerator of (1.1) are each sharp.

Let g be a metric (tensor) on $\mathbf{D}$ and let $g_0 = |dz|^2$ denote the euclidean metric. For a smooth, real-valued function ψ on $\mathbf{D}$ we define a symmetric, traceless 2-tensor

$$B_g(\psi) = \mathrm{Hess}_g\, \psi - d\psi \otimes d\psi - \frac{1}{2}\left\{\Delta_g \psi - \|\,\mathrm{grad}_g\, \psi\|_g{}^2\right\} g,$$

where, as we have indicated by the subscripts, the metric dependent quantities Hessian, gradient, Laplacian, and norm are computed with respect to g. (We use single bars, and no subscript, to denote the usual euclidean norm.) If $f\colon (\mathbf{D}, g) \to (\mathbf{C}, g_0)$ is a conformal, local diffeomorphism with $f^* g_0 = e^{2\psi} g$, its *Schwarzian tensor*, [19], [18], is defined by

$$\mathcal{S}_g f = B_g(\psi).$$

When g is the euclidean metric $\mathcal{S}_g f$ can be written as the matrix

$$\mathcal{S}_g f = (\mathrm{Re}\, Sf \; - \mathrm{Im}\, Sf \; - \mathrm{Im}\, Sf \; - \mathrm{Re}\, Sf).$$

For the arguments in this paper it will not be necessary to know all the aspects of this generalization of the Schwarzian. The familiar properties in the classical case are still present for the Schwarzian tensor, most importantly that

$$\mathcal{S}_g(M \circ f) = \mathcal{S}_g f \tag{1.3}$$

if M is a Möbius transformation. This is a special case of the chain rule

$$\mathcal{S}(f \circ h) = h^* \mathcal{S}(f) + \mathcal{S}h, \tag{1.4}$$

which includes the classical formula

$$S(f \circ h) = ((Sf) \circ h)(h')^2 + Sh.$$

The important thing to keep in mind is that the Schwarzian tensor is computed with respect to a background metric g, and it changes when g changes. When there is conformal change in g the Schwarzian tensor changes in a simple way, governed, in fact, by (1.4).

Here, as in the classical setting also, there are two very useful consequences of the Möbius invariance (1.3). First, so long as bounds on $\mathcal{S}_g f$ are unaffected, which will be the case in the situation we consider, it is possible in the course of a proof to normalize f in various ways by composing it with a Möbius transformation of the range. Second, one can also define the Schwarzian tensor for meromorphic functions by shifting the range of the function by a Möbius transformation in order to miss the point at infinity. We have some further comments on this, below.

In [18] the authors obtained a general univalence criterion in terms of $\mathcal{S}_g f$ that involves both the curvature of the metric and a diameter term. Let $K(g)$ denote the Gaussian curvature of the metric g. In the two-dimensional case the result can be stated as:

Theorem 1 *Let f be analytic or meromorphic in $(\mathbf{D}, g)$ and locally univalent. Suppose that any two points in $\mathbf{D}$ can be joined by a geodesic of length $< \delta$, for some $0 < \delta \leq \infty$. If*

$$||\mathcal{S}_g f||_g \leq \frac{2\pi^2}{\delta^2} - \frac{1}{2}K(g) \tag{1.5}$$

then f is univalent.

In [18] the formulation of this theorem is in terms of a conformal, local diffeomorphism of an n-dimensional Riemannian manifold, $n \geq 2$, into the n-sphere with its standard metric. In many ways having the sphere as the target is the most natural set-up. Adopting it in the two-dimensional case would allow us to dispense with the distinction between analytic and meromorphic functions, for instance. To make the tie-in with more classical results clearer, especially in defining extensions of the mapping, we decided to stick with the complex plane with its euclidean metric as the target. In any event, it makes no substantial difference in any of our results for the following reason. If f is a conformal, local diffeomorphism into either $\mathbf{C}$ with the euclidean metric or S^2 with its standard metric ($\widehat{\mathbf{C}}$ with the spherical metric), then, although the conformal factors are different under the pullback f^*, the Schwarzian tensor $\mathcal{S}_g f$ is the same in both cases. For this fact, see [19].

Many known criteria for univalence follow from Theorem 1 simply by choosing different conformal background metrics g. For example, as was pointed out in [18], if g is the euclidean metric, with $K = 0$ and $\delta = 2$, then (1.5) reduces to (1.2), while if g is $|dz|^2/(1-|z|^2)^2$, the Poincaré metric, with $K = -4$ and $\delta = \infty$, then one obtains the condition (1.1).

Similarly, one can obtain the very general criterion of Epstein, [14]:

$$\left| Sf(z) - 2(\tau_{zz} - \tau_z^2)(z) + \frac{4\bar{z}\tau_z(z)}{1-|z|^2} \right| \leq \frac{2(1 + (1-|z|^2)^2 \tau_{z\bar{z}}(z))}{(1-|z|^2)^2}. \tag{1.6}$$

In this case the metric to take in Theorem 1 is $e^{2\tau}|dz|^2/(1-|z|^2)^2$, where τ is a real-valued function satisfying some mild extra conditions. See [6] for the approach to Epstein's theorem using Theorem mail1, and [4] for an extension of (1.6) allowing for complex parameters.

Metrics on D and Associated Functions Unless noted otherwise, in the remainder of this paper we will *always assume* that

$$K(g) \leq 0,$$

and so we will not state this as a separate assumption in any of our results. Geometrically, the main consequence of this is that geodesics cannot cross more than once in $\mathbf{D}$.

If the metric g on the disk is complete we must take $\delta = \infty$ in Theorem 1. Then (1.5) becomes $||\mathcal{S}_g f||_g \leq -(1/2)K(g)$, which we will write as

$$||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)|. \tag{1.7}$$

In some instances, hypotheses, theorems, or proofs are different according to whether $\delta < \infty$ or $\delta = \infty$. For short we refer to the latter as 'the complete case'. (One can have $\delta = \infty$ but g not complete. We do not consider this case.)

We consider metrics on $\mathbf{D}$ of the form

$$g = e^{2\sigma}|dz|^2 = e^{2\sigma} g_0. \tag{1.8}$$

We let l_g denote the length function (of a curve) and d_g the distance (between points).

We recall that the curvature is given in terms of σ by

$$K(g) = -e^{-2\sigma}\Delta_{g_0}\sigma. \tag{1.9}$$

Using (1.4), (1.9), and $||\cdot||_g = e^{\sigma}|\cdot|$, the basic inequality on the Schwarzian, $||\mathcal{S}_g f||_g \leq 2(\pi/\delta)^2 - (1/2)K(g)$, in Theorem 1 can be written in euclidean terms as

$$|Sf - 2(\sigma_{zz} - \sigma_z^2)| \leq \frac{2\pi^2}{\delta^2}e^{2\sigma} + 2\sigma_{z\bar{z}}. \tag{1.10}$$

Let f be a conformal, local diffeomorphism of $(\mathbf{D}, g)$ into $(\mathbf{C}, g_0)$. Denoting

$$\varphi = \log|f'|, \tag{1.11}$$

we have

$$f^*g_0 = e^{2(\varphi-\sigma)}g, \tag{1.12}$$

and hence

$$\mathcal{S}_g f = B_g(\varphi - \sigma). \tag{1.13}$$

Definition We define

$$u_f = e^{(\sigma-\varphi)/2}. \tag{1.14}$$

We refer to u_f as the *associated function.*

If we use the euclidean metric in both the domain and the range of f then $u_f = |f'|^{-1/2}$. When the context is clear we write u for u_f.

We will use the function u_f throughout this paper. The basis of much of our analysis is the fact that there is a lower bound for the Hessian of u_f when f satisfies (1.5). This is Theorem 2 in Section 2. One then obtains bounds for u_f by means of comparison theorems for differential equations. In the complete case the result is that u_f is a convex function on $\mathbf{D}$ with respect to the metric g. In fact, the convexity of u_f becomes a characteristic property of functions satisfying (1.7) if one allows for composing f with a Möbius transformation of its range. This is Corollary 2 in Section 2.

If f is meromorphic in the disk then u_f is zero at a pole. At a pole u_f is not differentiable, so convexity, as a property of the Hessian, means convex away from the poles.

Boundary Behavior and Extremal Functions To study boundary behavior we need some special, global properties of the metric $g = e^{2\sigma}|dz|^2$ on $\mathbf{D}$. Here we make contact with the subject of 'visibility manifolds', an area of differential geometry that has been studied extensively. Of the literature on the subject we mention only the lectures of Eberlein [12] for a general survey of the early work, and a paper of Epstein [13] which is more directly related to the present paper.

The first property has to do with extending geodesics to the boundary, and with reaching every boundary point in this way. We state the property first as it often appears in the literature, but we must then say more to distinguish the complete and the non-complete cases.

Definition The metric g on $\mathbf{D}$ has the *Unique Limit Point* property (ULP) if:
(a) Let $z_0 \in \mathbf{D}$. If $\gamma(t)$, $0 \le t < T \le \infty$ is a maximally extended geodesic starting at z_0 then $\lim_{t \to T} \gamma(t)$ exists (in the euclidean sense). We denote it by $\gamma(T) \in \partial\mathbf{D}$.
(b) The limit point is a continuous function of the initial direction at z_0.
(c) Let $\zeta \in \partial\mathbf{D}$. Then there is a geodesic starting at z_0 whose limit point on $\partial\mathbf{D}$ is ζ.

We say a little more about part (c) in this condition. The assumption of nonpositive curvature implies that the limit point is a monotonic function of the initial direction at the base point. Part (*b*) requires that it is continuous. It is conceivable that, for some metrics, all geodesics from a base point might tend to the same limit point on the boundary, so the mapping from initial directions to points on $\partial\mathbf{D}$ would reduce to a constant. We want to avoid this degenerate situation and be certain that every boundary point is 'visible', so we include that fact in the statement of ULP.

(ULP) is a natural condition on complete metrics and is frequently formulated this way, if not with this appellation. For our work on boundary behavior in the non-complete case we have to strengthen it slightly. Again take any base point $z_0 \in \mathbf{D}$ and consider geodesics from z_0 extended maximally to their unique limit points on the boundary. In general, the length of such a geodesic as a function of the initial direction at z_0 is lower semicontinuous, and for our arguments we need to know that it is continuous. We let (ULP*) mean (ULP) *plus* the continuity of the length function. This is the assumption we will often adopt in the non-complete case. In the complete case the length function is the constant function $+\infty$ and the particular problems we encounter in the non-complete case do not come up; (ULP) will suffice as is.

The second global property we need is

Definition The metric g on $\mathbf{D}$ has the *Boundary Points Joined* property (BPJ) if any two points on $\partial\mathbf{D}$ can be joined by a geodesic which lies in $\mathbf{D}$ except for its endpoints.

The conditions above must be hypotheses in many of our results, but none of them, alone or together, is asking too much of a metric. Nevertheless, we need to know when they hold. In Section 6, we establish several conditions on the conformal factor σ implying the (ULP) *et al* conditions.

The fundamental result on univalence criteria and boundary behavior is Theorem 3 in Section 3, stating that when (ULP) or (ULP*) holds, a function satisfying (1.5) has a spherically continuous extension to $\overline{\mathbf{D}}$. This had been proved for functions satisfying Nehari's criterion (1.1) by Gehring and Pommerenke in [16].

We now make the following definition.

Definition Suppose the metric g satisfies (ULP), or (ULP*), and (BPJ). An analytic function f in $(\mathbf{D}, g)$ satisfying (1.5) is an *extremal function* for (1.5) if the extension of f to $\overline{\mathbf{D}}$ is not injective on $\partial\mathbf{D}$. A geodesic γ in $\mathbf{D}$ an *extremal geodesic* if it joins two points on $\partial\mathbf{D}$ where an extremal function f fails to be injective.

In Section 4 we study extremal functions and extremal geodesics in some detail. We show that equality holds in (1.5) along an extremal geodesic, and we prove an 'Image Circle Theorem', stating that the image of an extremal geodesic is a euclidean circle. This surprising geometric phenomenon was first discovered by Epstein [15] for his univalence criterion (1.6); it is essentially included in the case $\delta = \infty$ of our result. His methods were much different and do not apply to the case $\delta < \infty$.

Homeomorphic and Quasiconformal Extension There are strong forms of the univalence criteria (1.1), (1.6) and (1.7) having to do with quasiconformal extensions. Thus if the right hand side of the inequalities is multiplied by t, for $0 \le t < 1$, then the function f has a $\frac{1+t}{1-t}$-quasiconformal extension to $\widehat{\mathbf{C}}$. See [3], [14] and [6]. In this case one says that the image $\Omega = f(\mathbf{D})$ is a *quasidisk*.

In general, if f satisfies (1.7) then the image will not be a quasidisk, though it may be a Jordan domain. In Section 5 we address the question of constructing homeomorphic extensions to $\widehat{\mathbf{C}}$ of functions satisfying (1.7) under the assumption that the image is Jordan. We are able to find several characteristic properties for a function to satisfy (1.7), and also for $f(\mathbf{D})$ to be a Jordan domain. The result on homeomorphic extensions, together with the description of extremal functions and geodesics, can be viewed as a description of the possible degeneration that a quasiconformal extension can undergo as $t \to 1$. We can do this only in the complete case, and it is an interesting question to construct homeomorphic and quasiconformal extensions for functions satisfying a stronger form of (1.5) when $\delta < \infty$. For example, it follows from the work of Gehring and Pommerenke in [16] that the stronger form $|Sf| \leq t(\pi^2/2)$ of (1.2) implies that f has a quasiconformal extension. In [7] we are able to give an explicit formula for the extension in this case, but we cannot yet do so in general.

Example Finally, in Section 7 we apply our work to one particular example of a univalence criterion similar to one considered by Ahlfors [2]. A more detailed study of this example is presented in [10].

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2 Bounds on the Hessian, Convexity, and Critical Points

We begin with a computation relating the basic upper bound (1.5) on the Schwarzian to a lower bound for the Hessian of the associated function u_f defined in (1.14). We use this result for much of our analysis.

Theorem 2 *If* $||\mathcal{S}_g f||_g \leq \dfrac{2\pi^2}{\delta^2} - \dfrac{1}{2}K(g)$ *then* $\mathrm{Hess}_g\, u_f + \dfrac{\pi^2}{\delta^2} u_f\, g \geq 0.$

Proof. This actually follows from some of the computations in [18], but we give a direct verification here. Write u for u_f, and let $v = u^2 = e^{\sigma - \varphi}$. Then (see [19] or [18]),

$$\mathrm{Hess}_g\, v + v\mathcal{S}_g f = \frac{1}{2}(\Delta_g v)g, \tag{2.1}$$

We also have that

$$\Delta_g v = v\Delta_g(\log v) + \frac{1}{v}||\,\mathrm{grad}_g\, v||_g^2.$$

Using $\Delta_g = e^{-2\sigma}\Delta_{g_0}$ and $K(g) = -e^{-2\sigma}\Delta_{g_0}\sigma$ we obtain

$$\Delta_g v = ve^{-2\sigma}\Delta_{g_0}(\sigma - \varphi) + \frac{1}{v}||\,\mathrm{grad}_g\, v||_g^2 = -vK(g) + \frac{1}{v}||\,\mathrm{grad}_g\, v||_g^2.$$

Hence in (2.1),

$$\text{(2.2)} \qquad \begin{aligned} \mathrm{Hess}_g\, v &= v(-\frac{1}{2}K(g)g - \mathcal{S}_g f) + \frac{1}{2v}||\,\mathrm{grad}_g\, v||_g{}^2 g \\ &\geq -\frac{2\pi^2}{\delta^2} v\, g + \frac{1}{2v}||\,\mathrm{grad}_g\, v||_g{}^2 g. \end{aligned}$$

On the other hand, since $v = u^2$,

$$\mathrm{Hess}_g\, v = 2u\,\mathrm{Hess}_g\, u + 2du \otimes du, \quad \text{and} \quad \frac{1}{2v}||\,\mathrm{grad}_g\, v||_g{}^2 = 2||\,\mathrm{grad}_g\, u||_g{}^2.$$

It follows that

$$u\,\mathrm{Hess}_g\, u + du \otimes du \geq -\frac{\pi^2}{\delta^2}u^2 g + ||\,\mathrm{grad}_g\, u||_g{}^2 g,$$

so

$$\mathrm{Hess}_g\, u + \frac{\pi^2}{\delta^2} u\, g \geq 0,$$

as desired.

If $\gamma(t)$ is a unit-speed geodesic for g and $U(t) = u_f(\gamma(t))$, then along γ the inequality for the Hessian becomes

$$U'' + \frac{\pi^2}{\delta^2} U \geq 0.$$

Equality holds along γ if and only if equality holds in (1.5) along γ.

Next, a real-valued function w on $\mathbf{D}$ is convex with respect to g if the Hessian of w, computed with respect to g, is positive semi-definite. This is equivalent to requiring that $(w \circ \gamma)''(t) \geq 0$ for every geodesic $\gamma = \gamma(t)$ in $\mathbf{D}$, where t is an arclength parameter for g. When g is complete Theorem 2 is thus a convexity result. We use this often enough to merit a separate statement. (Recall that if f is meromorphic then u_f is zero at the pole. The computation in Theorem 2 applies away from the pole.)

Corollary 1 *If* $||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)|$ *then* u_f *is* g*-convex.*

From Corollary 1 we deduce a characterization of functions satisfying (1.7). Because the characterization involves shifting the range by an arbitrary Möbius transformation, the hypothesis is that f is meromorphic; see also [11].

Corollary 2 *Let* g *be complete and* f *a meromorphic function in* $(\mathbf{D}, g)$*. The following are equivalent:*
(a) $||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)|$;
(b) $u_{M\circ f}$ *is convex for all Möbius transformations* M*;*
(c) for every $z_0 \in \mathbf{D}$ *there exists a Möbius transformation* M *such that* $u_{M\circ f}$ *has a positive local minimum at* z_0*.*

Proof. *(a)* $\Rightarrow$ *(b)*: Since $\mathcal{S}_g(M \circ f) = \mathcal{S}_g f$ for any Möbius transformation M it suffices to show that u_f is convex, and this is precisely Corollary 1.
(b) $\Rightarrow$ *(c)*: Let $z_0 \in \mathbf{D}$. Since $u_{M\circ f}$ is convex, it suffices to choose M so that $u_{M\circ f}$ has a critical point at z_0, and z_0 is not a pole of f. But it is easy to see that an arbitrary Möbius transformation M has enough parameters to produce such a critical point.

$(c) \Rightarrow (a)$: Let $z_0 \in \mathbf{D}$ and suppose that $u = u_{M \circ f}$ has a positive local minimum at z_0. Then z_0 is not a pole of $M \circ f$, and, with $v = u^2$, from (2.2)

$$(\mathrm{Hess}_g\, v)(z_0) = v(-\frac{1}{2}K(g)g - \mathcal{S}_g(M \circ f))(z_0),$$

which must be ≥ 0. It follows that $||\mathcal{S}_g(M \circ f)||_g \leq -\frac{1}{2}K(g) = \frac{1}{2}|K(g)|$ at z_0. Since $\mathcal{S}_g(M \circ f) = \mathcal{S}_g f$, and since z_0 was arbitrary, the bound must hold everywhere.

Though Theorem 2 and Corollary 1 are local, under the assumption of completeness Corollary 1 has a useful global consequence based on the fact that a critical point of a smooth convex function is always a global minimum.

Corollary 3 *Let g be complete and f a meromorphic function in $(\mathbf{D}, g)$. If u_f has a critical point at which u_f is positive, then f is analytic. If the critical point is unique then f is bounded.*

Proof. For the first part, once again, at a pole of f the function u_f must vanish. This would then give a global minimum of u_f distinct from the one at the supposed critical point.

For the second part, let $z_0 \in \mathbf{D}$ be the unique critical point of $u = u_f$, $u(z_0) > 0$. Then $u(z_0)$ is the absolute minimum of u in $\mathbf{D}$.

We use geodesic polar coordinates r, θ on $\mathbf{D}$ based at z_0. Because the critical point is unique, given an $r_0 > 0$ there exists a $c > 0$ such that the radial derivative u_r is $\geq c$ at points z of g-distance $\geq r_0$ from z_0. Let $\gamma = \gamma(t)$ be a geodesic with $\gamma(0) = z_0$, and write $U(t) = u(\gamma(t))$. Then $U'(t) \geq c$ for $t \geq r_0$, and since $U(t)$ is convex

$$U(t) \geq b + ct, \tag{2.3}$$

for some constant b independent of θ. Hence

$$e^{\varphi(\gamma(t))}e^{-\sigma(\gamma(t))} = U^{-2}(t) \leq \frac{1}{(b+ct)^2}, \tag{2.4}$$

and so for all $T > r_0$,

$$\begin{aligned}\int_{\gamma|_{[r_0,T]}} |f'(z)|\,|dz| &= \int_{r_0}^{T} e^{\varphi(\gamma(t))}e^{-\sigma(\gamma(t))}\,dt \\ &\leq \int_{r_0}^{T} \frac{1}{(b+ct)^2}\,dt \leq \int_{r_0}^{\infty} \frac{1}{(b+ct)^2}\,dt < \infty.\end{aligned}$$

Since g is complete, $\gamma(T) \to \partial\mathbf{D}$ as $T \to \infty$, and we conclude that $f(\mathbf{D})$ is bounded.

Both the statement of this corollary and its proof will be used in later arguments.

Remark. Corollary 3 is a distortion theorem in disguise. We can always assume that the critical point of u_f is at the origin by changing f to $M \circ f$ by a Möbius transformation M, and we can normalize further so that $u_f(0) = 1$. Even if the critical point is not unique the convexity of u_f implies that $e^{(\sigma-\varphi)/2} = u_f(z) \geq u_f(0) = 1$, or

$$|f'| = e^{\varphi} \leq e^{\sigma}.$$

When g is the Poincaré metric with $e^{\sigma(z)} = 1/(1-|z|^2)$ this becomes

$$|f'(z)| \leq \frac{1}{1-|z|^2},$$

which is the sharp upper bound for functions $f(z) = z + a_3 z^3 + \cdots$ satisfying Nehari's condition $|Sf(z)| \le 2/(1-|z|^2)^2$, [8]. When the critical point is unique, (2.3) implies that for $R_0 \le |z| < 1$ one has

$$|f'(z)| \le \frac{e^{\sigma(z)}}{(b + c d_g(0,z))^2}, \tag{2.5}$$

where now the constants R_0, b and c depend on f. Here d_g denotes the distance in the g-metric. In some cases one can deduce estimates for the modulus of continuity from (2.5).

We will make further use of convexity and critical points in Section 5. Analogous results on distortion for the non-complete case have eluded us.

3 Extension to $\overline{\mathbf{D}}$

In this section we use the differential inequality provided by Theorem 2 to prove that a function satisfying the general univalence criterion (1.5) has a continuous extension to $\overline{\mathbf{D}}$. Here, for the first time, we we must assume the unique limit point property (ULP).

Theorem 3 *Suppose f is a meromorphic function in $(\mathbf{D}, g)$ satisfying* (1.5)*, and that g satisfies (ULP) if it is complete and (ULP*) if it is not complete. Then f admits a (spherically) continuous extension to $\overline{\mathbf{D}}$.*

Proof. Let $\Omega = f(\mathbf{D})$. We will show that small arcs on S^1, corresponding to intervals of initial directions of geodesics from a base point, parametrize small arcs on $\partial\Omega$. This implies that $\partial\Omega$ is locally connected at each point, which is a necessary and sufficient condition for f to have a continuous extension to $\overline{\mathbf{D}}$. To obtain the requisite estimates we have to modify f by Möbius transformations of the range, and this is why the theorem is stated in terms of meromorphic rather than analytic functions.

The proof is slightly different in the two cases $\delta < \infty$ and $\delta = \infty$. We consider first $\delta < \infty$; thus (ULP*) is in force. Let $\zeta_0 \in \partial\mathbf{D}$ and let γ_0 be a geodesic in $\mathbf{D}$ ending at ζ_0. Let $z_0 \in \gamma_0$ be a point of distance $< \delta/8$ from ζ_0, and let θ_0 be the direction of γ_0 at z_0. Choose a small enough neighborhood V of initial directions about θ_0 with corresponding geodesics covering an arc $I \subset \partial\mathbf{D}$ of limit points so that the distances between z_0 and all such limit points is $\le \delta/4$.

Let $\theta \in V$ and let $\gamma(t)$, $0 \le t \le T_\theta$ be the corresponding geodesic starting at z_0 and ending at a point on $I \subset \partial\mathbf{D}$. Replace f by $M \circ f$, where the Möbius transformation M is chosen so that the associatied function $u_{M\circ f}$ satisfies

$$\operatorname{grad} u_{M\circ f}(z_0) = 0 \quad \text{and} \quad u_{M\circ f}(z_0) = 1.$$

We want to apply Theorem 2 to $u_{M\circ f}$ along the geodesics γ. Since $\mathcal{S}_g(M \circ f) = \mathcal{S}_g f$, we continue to write f for $M \circ f$ and u_f for $u_{M\circ f}$. The function $U(t) = u_f(\gamma(t))$ satisfies

$$U'' \ge -\frac{\pi^2}{\delta^2} U, \quad U(0) = 0, \quad U'(0) = 1.$$

From this,

$$U(t) \ge \cos(\frac{\pi}{\delta} t),$$

and so

$$U(t) \geq \cos(\frac{\pi}{\delta}\frac{\delta}{4}) = \frac{1}{\sqrt{2}}.$$

Note that since u_f is non-zero in the sector swept out by the geodesics γ, f is analytic there. Referring to (1.11) and (1.14),

$$|f'| = e^{\varphi} \leq 2e^{\sigma},$$

along γ, and

$$\int_{\gamma} |f'|\,|dz| \leq 2l_g(\gamma) \leq \frac{\delta}{2}. \tag{3.1}$$

This implies that

$$\lim_{t\to T_\theta} f(\gamma(t))$$

exists. We denote the limit by $f(\gamma(T_\theta))$; it lies on $\partial\Omega$.

We prove next that $f(\gamma(T_\theta)) \in \partial\Omega$ depends continuously on the initial direction θ of the geodesic. Let γ_1, $0 \leq t \leq T_{\theta_1}$ and γ_2, $0 \leq t \leq T_{\theta_2}$, be two geodesic rays starting at z_0 with $\theta_1, \theta_2 \in V$. We need to estimate the distance between $f(\gamma_1(T_{\theta_1}))$ and $f(\gamma_2(T_{\theta_2}))$. Let $0 < \tau < \min\{T_{\theta_1}, T_{\theta_2}\}$. Then

$$\begin{aligned}|f(\gamma_1(T_{\theta_1})) - f(\gamma_2(T_{\theta_2}))| \leq{} & |f(\gamma_1(T_{\theta_1})) - f(\gamma_1(\tau))| + |f(\gamma_1(\tau)) - f(\gamma_2(\tau))| \\ & + |f(\gamma_2(T_{\theta_2})) - f(\gamma_2(\tau))|.\end{aligned}$$

The terms $|f(\gamma_i(T_{\theta_i})) - f(\gamma_i(\tau))|$ are dominated by the tails of the integrals in (3.1) which are uniformly bounded by $\delta/2$. Now using the continuity of the length function in the hypothesis (ULP*), there is a τ_0 so that both these terms are small for $\tau_0 \leq \tau < \min\{T_{\theta_1}, T_{\theta_2}\}$ if $|\theta_1 - \theta_2|$ is small. The remaining term can be controlled using the continuity of f and the fact that $|\gamma_1(\tau) - \gamma_2(\tau)|$ is small if $|\theta_1 - \theta_2|$ is small. These estimates prove that the endpoints $f(\gamma(T_\theta)) \in \partial\Omega$, γ varying, depend continuously on the initial directions $\theta = \gamma'(0)$.

It remains to show that any point in $\partial\Omega$ is the image $f(\gamma(T_\theta))$ as in the construction above. Let $\omega \in \partial\Omega$ and let $\{w_n\}$ be a sequence of points in Ω which converges to ω. Choose a subsequence, labeled the same way, of $z_n = f^{-1}(w_n)$ converging to a point $\zeta \in \partial\mathbf{D}$. Let $z_0 \in \mathbf{D}$ be a point of distance $< \delta/8$ from ζ.

Let g_1 be the metric on Ω obtained by pulling back the metric g on $\mathbf{D}$ by f^{-1}. Thus $f\colon (\mathbf{D}, g) \to (\Omega, g_1)$ is an isometry. Let let $\Gamma_n(t)$ be the g_1-geodesic joining $f(z_0) = w_0$ to w_n with $\Gamma_n(0) = w_0$. Another subsequence, again labeled in the same way, of the initial directions $\Gamma_n'(0)$ converges to a direction which determines a geodesic Γ. Let $\gamma = f^{-1}(\Gamma)$, $\gamma = \gamma(t)$, $\theta = \gamma'(0)$, $0 \leq t \leq T_\theta$. Let $\gamma_n = f^{-1}(\Gamma_n)$ and let $t_n = l_g(\gamma_n) = l_{g_1}(\Gamma_n)$. Write

$$\begin{aligned}|f(\gamma(T_\theta)) - w_n| &= |f(\gamma(T_\theta)) - f(\gamma_n(t_n))| \\ &\leq |f(\gamma(T_\theta)) - f(\gamma(\tau))| + |f(\gamma(\tau)) - f(\gamma_n(\tau))| \\ &\quad + |f(\gamma_n(\tau)) - f(\gamma_n(t_n))|.\end{aligned}$$

As $\gamma_n'(0) \to \gamma'(0) = \theta$, we conclude for n sufficiently large that $|f(\gamma(T_\theta)) - w_n|$ can be made arbitrarily small by choosing τ close enough to T_θ. Hence $\omega = f(\gamma(T_\theta))$. This completes the proof in the case $\delta < \infty$.

We indicate now how the argument should be modified in the complete case $\delta = \infty$. Choose a base point z_0, which is fixed for the entire argument. Let $w_0 = f(z_0)$. The g_1-geodesic rays from w_0 can be extended indefinitely, and we need to know that they have a limit. Any such ray is the image under f of a geodesic $\gamma = \gamma(t)$, $\gamma(0) = z_0$. Changing f by an appropriate Möbius transformation of the range, and maintaining the same notation convention as above, we may assume that $U'(0) \geq c > 0$. Then, as in the proof of Corollary 3, we have $U(t) \geq b + ct$, $t \geq 0$, and

$$\int_\gamma |f'|\,|dz| < \infty. \tag{3.2}$$

Thus $\lim_{t\to\infty} f(\gamma(t))$ exists, and we denote if by $f(\gamma(\infty)) \in \partial\Omega$.

For the continuity of $f(\gamma(\infty))$ depending on the initial directions at z_0 we argue as follows. Take a geodesic $\gamma_1(t)$ from z_0. This time we modify f by a Möbius transformation to change the gradient of u_f at z_0 so that $U'(0) \geq c > 0$ for all rays from z_0 that form an angle of less than $\pi/4$ with $\gamma_1'(0)$. This makes the integrals in (3.2) uniformly bounded over all such rays, and f uniformly bounded in the sector covered by the rays. From here the proof of continuity, and that all of $\partial\Omega$ is hit by the $f(\gamma(\infty))$, is almost identical to the above. Only (ULP) is necessary.

4 Extremal Functions

Suppose the metric g satisfies (ULP), or (ULP*), and (BPJ). Recall that f is an extremal function for (1.5) if $f(\mathbf{D})$ is not a Jordan domain, and that a corresponding extremal geodesic joins two points on $\partial\mathbf{D}$ where f is not injective.

The principal result on extremal functionas and extremal geodesics is the following.

Theorem 4 *Let g have the properties (ULP), (or (ULP*)) and (BPJ). Then*
(i) Equality holds in (1.5) *for an extremal function along an extermal geodesic.*
(ii) The image $f(\gamma)$ of an extremal geodesic under the extremal function f is a euclidean circle.

Part (i) of this theorem is another reason for the term extremal function. Its converse, however, is not true. Take Nehari's criterion $|Sf(z)| \leq 2/(1-|z|^2)^2$. The interval $(-1,1)$ is an extremal geodesic for the function

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}.$$

But we also have $|Sf(z)| = 2/(1-|z|^2)^2$ along $(-1,1)$ for the function

$$f(z) = \frac{1}{\sqrt{2}}\,\frac{(1+z)^{\sqrt{2}} - (1-z)^{\sqrt{2}}}{(1+z)^{\sqrt{2}} + (1-z)^{\sqrt{2}}}, \quad Sf(z) = \frac{-2}{(1-z^2)^2},$$

and $f(\mathbf{D})$ is a Jordan domain, in fact a quasidisk. Hence, in our sense, f is not an extremal function for Nehari's criterion.

Let f be an extremal function for (1.5), with $f(\zeta_1) = f(\zeta_2)$, and let γ be an extremal geodesic joining ζ_1 and ζ_2. Theorem 4 does not assume any normalizations on f, but the proof, which depends on properties of the associated function, needs some. Normalize f via $M \circ f$, where M is a Möbius transformation, so that

$$f(\zeta_1) = f(\zeta_2) = \infty. \tag{4.1}$$

Let $\gamma(t)$, be a g-unit speed parametrization of γ in the direction from ζ_2 to ζ_1. We consider the associated function u_f restricted to an extremal geodesic γ. As earlier we let

$$U(t) = u_f(\gamma(t)).$$

For the complete case we have the following preliminary result.

Lemma 1 *If $\delta = \infty$ then $U(t)$ is constant.*

Proof. Recall from Corollary 1 that $U(t)$ is a convex function of t. If $U(t)$ were not constant it would be bounded from below by some nonconstant affine function $b + ct$, as in the proof of Corollary 3. But then this would make one of the integrals

$$\int_{\gamma^+} |f'(z)|\,|dz|, \qquad \int_{\gamma^-} |f'(z)|\,|dz|$$

finite, where $\gamma^+ = \gamma|_{[0,\infty)}$ and $\gamma^- = \gamma|_{(-\infty,0]}$. This contradicts $f(\zeta_1) = f(\zeta_2) = \infty$.

Later we will show more precisely that the constant value of U is the absolute minimum of u_f on $\mathbf{D}$.

For the case $\delta < \infty$ there are two basic lemmas. We maintain the nomalization (4.1).

Lemma 2 *Suppose $\delta < \infty$. Then an extremal geodesic has length δ and its midpoint is the unique critical point of $U(t)$.*

Proof. We show first that $U(t)$ must have a critical point. If not then U is monotone, say increasing. Consequently $U(t) \geq U(t_0) = a$ for $t \geq t_0$, and thus

$$|f'(z)| \leq \frac{1}{a^2} e^{\sigma(z)}$$

for $z \in \gamma$ after $\gamma(t_0)$. This gives that

$$\int_{\gamma(t_0)}^{\zeta_1} |f'|\,|dz| \leq \frac{\delta}{a^2} < \infty,$$

and therefore that $f(\zeta_1)$ is finite, contradicting the normalization of f.

Let $z_0 = \gamma(0)$ be a critical point for $U(t)$. Since $U'' \geq -(\pi/\delta)^2 U$, it follows that

$$U(t) \geq U(0)\cos(\frac{\pi}{\delta}t) \tag{4.2}$$

for $|t| < \delta/2$. If either $d_g(z_0, \zeta_1)$ or $d_g(z_0, \zeta_2)$ is $< \delta/2$ then u would be bounded below by a positive constant on either the part of γ from z_0 to ζ_1 or from ζ_2 to z_0. As before, this leads to a contradiction with the normalization of f.

Since in any case $d_g(z_0, \zeta_1) + d_g(z_0, \zeta_2) \leq \delta$, we conclude that $d_g(z_0, \zeta_1) = d_g(z_0, \zeta_2) = \delta/2$. This also shows that the critical point is unique.

Lemma 3 *If $\delta < \infty$ and $z_0 = \gamma(0)$ is the midpoint then*

$$U(t) = U(0)\cos(\frac{\pi}{\delta}t), \quad -\frac{\delta}{2} \leq t \leq \frac{\delta}{2}. \tag{4.3}$$

Proof. As in (4.2), we have

$$U(t) \geq U(0)\cos(\frac{\pi}{\delta}t), \quad -\frac{\delta}{2} \leq t \leq \frac{\delta}{2},$$

hence $U(t) > 0$. We also know that U is monotone for $-\delta/2 < t < 0$ and for $0 < t < \delta/2$. We claim it is increasing on the negative interval and decreasing on the positive. Suppose not, say that U is increasing for $0 \leq t \leq \delta/2$. Then $U(t) \geq U(0) = a$ for $0 < t < \delta/2$, which implies as before that $f(\zeta_1)$ is finite, a contradiction.

Since $U > 0$, we conclude that both limits $\lim_{t\to\pm\delta/2} U(t)$ exist. But these limits must be zero, for otherwise U would be bounded from below by a positive constant on some half of γ. Hence $U(\delta/2) = U(-\delta/2) = 0$ and again the Sturm comparison theorem implies (4.3).

Part (i) of Theorem 4 now follows from these lemmas. For in either of the cases $\delta < \infty$ or $\delta = \infty$ the function U satisfies

$$U'' + \frac{\pi^2}{\delta^2}U = 0$$

along the extremal geodesic, and this implies that equality holds in (1.5) there. Since $\mathcal{S}_g(M \circ f) = \mathcal{S}_g f$ for a Möbius transformation M, the same is true for any extremal function with this extremal geodesic, normalized or not.

We turn to the geometry of extremal geodesics. Let f be an extremal function for (1.5) with an extremal geodesic γ joining $\zeta_1, \zeta_2 \in \partial\mathbf{D}$ where $f(\zeta_1) = f(\zeta_2)$. Normalize f as in (4.1). To prove part (ii) of Theorem 4 we then want to show that $f(\gamma)$ is a straight line, and we do this by showing that its euclidean curvature is zero.

First we need a general formula. If $\xi\colon [a,b] \to \mathbf{C}$ is a curve with $\xi' \neq 0$ then the Schwarzian $S\xi$ is defined by the same formula as for analytic functions. When $\xi(s)$ is a (euclidean) arclength parametrization then $\xi'(s) = e^{i\theta(s)}$ and

$$\frac{\xi''}{\xi'} = i\theta' = ik,$$

where k is the curvature. Thus

$$S\xi = ik' + \frac{1}{2}k^2, \tag{4.4}$$

a well known formula, see for example [1] p. 21.

Next, when $\delta < \infty$ let z_0 be the g-midpoint of γ, as in Lemma 3, and normalize f further so that

$$u_f(z_0) = 1. \tag{4.5}$$

It follows from Lemma 3 and the definition of u_f that, along γ,

$$|f'(z)| = e^{\sigma(z)}\cos^{-2}(\frac{\pi}{\delta}d_g(z, z_0)), \tag{4.6}$$

where d_g is the distance in the metric g. Though this formula is for the case $\delta < \infty$, (4.6) includes the complete case. That is, if $\delta = \infty$ then

$$|f'| = e^{\sigma} \tag{4.7}$$

along γ, where, using the result of Lemma 1, we further normalized f to have u_f identically 1 along γ.

Let $\xi_1 = \xi_1(s)$ be a euclidean arclength parametrization of γ. We introduce the following real-valued function, modeled on f along γ. Define $f_0(s)$ by

$$f_0'(s) = e^{\sigma(\xi_1(s))} \cos^{-2}(\frac{\pi}{\delta} d_g(\xi_1(s), z_0)), \quad f_0(0) = 0. \tag{4.8}$$

We need the fact that the Schwarzians of $f_0(s)$ and of $\xi_1(s)$ are related through

$$Sf_0 - 2(\sigma_{zz} - \sigma_z^2)(\xi_1')^2 = 2\frac{\pi^2}{\delta^2} e^{2\sigma} + 2\sigma_{z\bar{z}} + S\xi_1, \tag{4.9}$$

where in this equation and elsewhere in the proof σ and its z-derivatives are evaluated at $\xi_1(s)$.

To derive this, we first have

$$\frac{f_0''}{f_0'} = 2\,\mathrm{Re}\{\sigma_z \xi_1'\} + 2\frac{\pi}{\delta} e^{\sigma} \tan(\frac{\pi}{\delta} d_g),$$

where we have used that the derivative of $d_g(\xi_1(s), z_0)$ is 1 when differentiating along γ with respect to g-arclength, and hence is e^{σ} when differentiating with respect to the euclidean arclength s. Since γ is a g-geodesic its euclidean curvature is

$$k_1 = -2\,\mathrm{Im}\{\sigma_z \xi_1'\},$$

thus

$$\frac{f_0''}{f_0'} = 2\sigma_z \xi_1' + ik_1 + 2\frac{\pi}{\delta} e^{\sigma} \tan(\frac{\pi}{\delta} d_g).$$

Differentiate again and use $|\xi_1'| = 1$ and $\xi_1'' = -k_1 \xi_1'$. With a little effort this leads directly to (4.9).

Next, using (1.10), the euclidean form of (1.5), we appeal to part (i) of the Theorem 4 and observe that for an extremal f there is a function $\varepsilon(z)$ along γ with $|\varepsilon| = 1$ such that

$$Sf - 2(\sigma_{zz} - \sigma_z^2) = (\frac{2\pi^2}{\delta^2} e^{2\sigma} + 2\sigma_{z\bar{z}})\varepsilon, \tag{4.10}$$

along γ.

Now let $\xi_2 = \xi_2(t)$ be a euclidean arclength parametrization of the image curve $f(\gamma)$. Then by construction

$$f(\xi_1(s)) = \xi_2(f_0(s)).$$

Taking Schwarzians of both sides we obtain

$$(Sf)(\xi_1')^2 + S\xi_1 = (S\xi_2)(f_0')^2 + Sf_0,$$

which together with (4.10) and (4.9) gives

$$2(\frac{\pi^2}{\delta^2} e^{2\sigma} + \sigma_{z\bar{z}})(\xi_1')^2 \varepsilon = 2\frac{\pi^2}{\delta^2} e^{2\sigma} + 2\sigma_{z\bar{z}} + (S\xi_2)(f_0')^2.$$

The left hand side of (4) has absolute value $2(\pi/\delta)^2 e^{2\sigma} + 2\sigma_{z\bar{z}}$, while the right hand side will have the same absolute value if and only if $S\xi_2 = 0$. Recalling that $S\xi_2 = ik_2' + (1/2)k_2^2$, where k_2 is the euclidean curvature of $\xi_2 = f(\gamma)$, this implies that $f(\gamma)$ must be part of a straight line. But then it must be the entire straight line because both endpoints are at infinity.

This completes the proof of part (ii) of Theorem 4 when f is normalized, and hence in general.

Before continuing, we note that an extremal function f normalized as above, which maps an extremal geodesic γ to a straight line, is completely determined along γ. We know $|f'|$ along γ by (4.6) or (4.7), and if $f(\gamma)$ is a line we also know the argument. So, for instance, if the image is the real axis then for $z \in \gamma$ we have, up to a constant,

$$f(z) = \frac{\delta}{\pi} \tan(\frac{\pi}{\delta} d_g(z, z_0))$$

when $\delta < \infty$, and

$$f(z) = d_g(z, z_0),$$

when $\delta = \infty$. Here z_0 is the midpoint of γ in the first case and any fixed point on γ in the second. (Thus f is the developing map for the metric g along γ, for readers familiar with that terminology.)

We now deduce further properties of the associated function u_f along an extremal geodesic in the complete case.

Corollary 4 *Suppose g is complete and let f be an extremal function. Under the normalizations* (4.1) *the associated function satisfies* $\operatorname{grad} u_f = 0$ *along an extremal geodesic γ, and assumes its absolute minimum in* $\mathbf{D}$ *along γ.*

Proof Since $f(\gamma)$ is a straight line, we may rotate f if necessary and assume that $f(\gamma)$ is the real axis. Let $\xi = \xi(s)$ be a euclidean arclength parametrization for γ. Since $f(\gamma)$ is real, along γ we have $\arg f' = -\arg \xi'$. Thus

$$f'\xi' = e^{\sigma}. \tag{4.11}$$

From this,

$$\xi' \frac{f''}{f'} + \frac{\xi''}{\xi'} = 2\operatorname{Re}\{\sigma_z z_1'\}, \tag{4.12}$$

and using the equation for curvature, $\xi''/\xi' = ik = -2\operatorname{Im}\{\sigma_z \xi'\}$, we get

$$\frac{f''}{f'} = 2\sigma_z. \tag{4.13}$$

It is easy to see that this last equation is equivalent to $\operatorname{grad} u_f = 0$ along γ.

The function u_f is constant on γ by Lemma 1, and this value is the absolute minimum of u_f in $\mathbf{D}$ by convexity.

In Theorem 6 in the next section we will need to *prove* that a geodesic is extremal. Thus as a complement to the preceding results, we need the following elementary and general fact.

Lemma 4 *Let f satisfy* (1.5). *Suppose γ is a geodesic segment in* $\mathbf{D}$ *along which u_f attains its absolute minimum. Then $f(\gamma)$ is a straight line segment in $f(\mathbf{D})$.*

Proof. We are not assuming that the metric is complete, and in fact this is the one case where we do not need that the curvature is nonpositive.

Let $\Omega = f(\mathbf{D})$ and let g_1 be the pullback of g under f^{-1}. Thus $f\colon (\mathbf{D}, g) \to (\Omega, g_1)$ is an isometry. The metric g_1 is also conformal to the euclidean metric. For purposes fully explained in the next section, we write is as $g = \rho_f^2 |dw|^2$, so that $\rho_f \circ f = u_f^2$. Now, by hypothesis, ρ_f attains its absolute minimum along the g_1-geodesic $\Gamma = f(\gamma)$ in Ω. It is easy to see from this that the differential equation satisfied by Γ reduces to $d^2\Gamma/ds^2 = 0$.

5 Reflections and Extensions in the Complete Case

In this section we consider the problem of homeomorphic and quasiconformal extensions to $\widehat{\mathbf{C}}$ of functions satisfying

$$||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)| \tag{5.1}$$

when $g = e^{2\sigma}|dz|^2$ is complete. Suppose also that the metric satisfies (ULP) and (BPJ). Then such an f has a continuous extension to $\overline{\mathbf{D}}$, and if there is an extremal function there is also a corresponding extremal geodesic.

Let $\Omega = f(\mathbf{D})$. As on earlier occasions, we define a metric on Ω by

$$g_1 = (f^{-1})^* g,$$

and we write

$$g_1 = \rho_f^2 |dw|^2.$$

Then $f\colon (\mathbf{D}, e^{2\sigma}|dz|^2) \to (\Omega, \rho_f^2|dw|^2)$ is an isometry, and

$$\rho_f \circ f = u_f^2.$$

We define a mapping $\Lambda = \Lambda_f$ of Ω by

$$\Lambda(w) = w + \frac{1}{\partial_w \log \rho_f(w)}. \tag{5.2}$$

Under certain circumstances Λ_f will be a reflection across $\partial\Omega$, and will allow us to define an extension E_f of f.

We shall need a property of Λ_f known as *conformal naturality.*

Lemma 5 *If M is a Möbius transformation then $\Lambda_{M\circ f} = M \circ \Lambda_f$.*

The equation in the lemma means that

$$\Lambda_{M\circ f}(M(f(z))) = M(\Lambda_f(f(z))) \tag{5.3}$$

for any point $z \in D$.

Proof. The identity (5.3) is easy to check for a similarity, so we go through the calculation only for an inversion. With $h = 1/f$, we have $h' = -f'/f^2$ and from this,

$$(h^{-1})^* g = \tau^2 |dz|^2 \quad \text{with} \quad \tau = |f|^2 e^{\sigma - \varphi} = |f|^2 \rho_f,$$

or

$$\log \tau = \log \rho_f + 2 \log |f|.$$

With $w = f(z)$ and $\zeta = 1/w$ we now compute that

$$\partial_\zeta \log \tau = -w^2 \partial_w \log \rho_f - w.$$

Then

$$\zeta + \frac{1}{\partial_\zeta \log \tau} = \frac{1 + \frac{1}{w}(-w - w^2 \partial_w \log \rho_f)}{-w - w^2 \partial_w \log \rho_f} = \frac{\partial_w \log \rho_f}{1 + w \partial_w \log \rho_f} = \frac{1}{w + \dfrac{1}{\partial_w \log \rho_f}},$$

which is the desired identity.

The combination of the following two lemmas gives conditions for Λ_f to be a reflection across $\partial\Omega$.

Lemma 6 *If $u = u_f$ has a unique critical point in* $\mathbf{D}$ *then*

$$(5.4)\qquad |\partial_w \log \rho_f| \to \infty \quad as \quad w \to \partial\Omega.$$

Proof. We compute that

$$|\partial_w \log \rho_f| = \frac{1}{\rho_f}|\partial_w \rho_f| = u^{-2}e^{-\varphi}2u|\partial_z u| = 2ue^{-\sigma}|\partial_z u| = ue^{-\sigma}|\operatorname{grad}_{g_0} u|.$$

Using the general relations $\operatorname{grad}_g = e^{-2\sigma}\operatorname{grad}_{g_0}$ and $||\cdot||_g = e^{\sigma}|\cdot|$, we thus find that

$$(5.5)\qquad |\partial_w \log \rho_f| = u||\operatorname{grad}_g u||_g.$$

Suppose z_0 is the unique critical point of u in $\mathbf{D}$. Then as in the proof of Corollary 3, $||\operatorname{grad}_g u||_g \geq c > 0$ outside some compact set containing z_0. It follows from (5.5) and from (2.4) that $|\partial_w \log \rho|$ becomes unbounded near $\partial\Omega$, as desired.

Lemma 7 *Suppose $||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)|$, and that g is complete. Let $\Omega = f(\mathbf{D})$. Suppose for every Möbius transformation M that $u_{M\circ f}$ has at most one critical point in $\mathbf{D}$. Then Λ_f takes values in $\widehat{\mathbf{C}}\backslash\overline{\boldsymbol{\Omega}}$.*

Proof. Suppose to the contrary that there exists a $w_1 \in \Omega$ such that $\Lambda_f(w_1) \in \overline{\Omega}$. Choose a Möbius transformation M such that $u_{M\circ f}$ has a critical point at $z_1 = f^{-1}(w_1)$. By assumption, z_1 is therefore the unique critical point of $u_{M\circ f}$, and again by Corollary 3, $(M \circ f)(\mathbf{D}) = M(\Omega)$ is bounded. But $\Lambda_f(w_1) = \Lambda_f(f(z_1)) \in \overline{\Omega}$, hence $M(\Lambda_f(f(z_1))) \in \overline{M(\Omega)}$. On the other hand, by (5.3),

$$M(\Lambda_f(f(z_1))) = \Lambda_{M\circ f}(M(f(z_1))) = \infty,$$

the last equality because z_1 is a critical point for $u_{M\circ f}$. This contradicts the boundedness of $M(\Omega)$.

We now define an extension of f by the formula

$$(5.6)\qquad E_f(z) = f(z)\text{for } |z| \leq 1, \Lambda(f(1/\bar{z}))\text{for } |z| > 1.$$

Theorem 5 *Let g satisfy (ULP) and (BPJ). Suppose $||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)|$, and that g is complete. The following are equivalent:*
(a) E_f is a homeomorphism of $\widehat{\mathbf{C}}$;
(b) Λ_f is injective with values in $\widehat{\mathbf{C}}\backslash\overline{\boldsymbol{\Omega}}$;
(c) for each Möbius transformation M, $u_{M\circ f}$ has at most one critical point in $\mathbf{D}$.

Proof. $(a) \Rightarrow (b)$: This is an immediate consequence of the definition of E_f.
$(b) \Rightarrow (c)$: Because of the conformal naturality of the extension, the hypothesis in (b) is invariant under Möbius changes $M \circ f$. Observe that a critical point of u in $\mathbf{D}$ corresponds under f to a critical point of ρ_f in Ω, which is in turn mapped by Λ_f to the point at infinity. Hence u_f can have at most one critical point.
$(c) \Rightarrow (a)$: This implication is the core of the theorem, and by now most of the work is done. First, $f(\mathbf{D})$ must be a Jordan domain, for if not then f is an extremal function and there is an extremal geodesic, say joining $\zeta_1, \zeta_2 \in \partial\mathbf{D}$. We may assume that f is normalized so that $f(\zeta_1) = f(\zeta_2) = \infty$. Then $\operatorname{grad} u_f = 0$ along γ by Corollary 4. Next, by Lemma 6, E_f is continuous on $|z| = 1$, and

its (spherical) continuity elsewhere is clear. Finally, in order to show that E_f is globally injective it suffices to show that this is so for Λ_f, as Λ_f takes values outside $\overline{\Omega}$. Suppose $\Lambda_f(w_1) = \Lambda_f(w_2)$. Via a Möbius transformation we may assume that this common value is the point at infinity. In that case, the corresponding function u must have critical points at $f^{-1}(w_1)$ and $f^{-1}(w_2)$, hence $f^{-1}(w_1) = f^{-1}(w_2)$. Therefore $w_1 = w_2$. This proves that E_f is a continuous, injective map of $\hat{\mathbf{C}}$ onto itself, and hence is a homeomorphism.

We studied homeomorphic extensions in an earlier paper[1] [9] for the Ahlfors-Weill extension [3], which is precisely E_f when g is the Poincaré metric.

From the theorem we obtain a result on quasiconformal extension, requiring however that the curvature be strictly negative.

Corollary 5 *Let g satisfy (ULP) and (BPJ). If $K(g) < 0$ and if*

$$||\mathcal{S}_g f||_g \leq \frac{t}{2}|K(g)| \tag{5.7}$$

for some $0 \leq t < 1$, then f has a $\frac{1+t}{1-t}$-quasiconformal extension to $\hat{\mathbf{C}}$.

Proof. We claim first that E_f is a homeomorphic extension of f. Suppose by way of contradiction that there is some Möbius transformation M so that $u_{M\circ f}$ has at least two critical points in $\mathbf{D}$. Then by convexity $u_{M\circ f}$ attains its absolute minimum all along the geodesic segment between the two points. As in the last section this implies that $||\mathcal{S}_g f||_g = (1/2)|K(g)|$ along that segment, a contradiction.

Next, one computes as in [5] that the Beltrami coefficient μ of Λ_f has magnitude

$$|\mu \circ f| = \frac{2}{|K(g)|}||\mathcal{S}_g f||_g. \tag{5.8}$$

This is $\leq t < 1$ and the conclusion follows.

Remark. In terms of the classical Schwarzian, Corollary 5 is essentially the strong form of Epstein's univalence criterion (1.6) as given in [14]. The version here in terms of the Schwarzian tensor appears in [5] and [6]. The form (5.2) of the reflection Λ and (5.6) of the quasicionformal extension E_f are also Epstein's. His construction in [14], involving an ingenious use of reflections in surfaces in hyperbolic space, is much different from the one given here. (The conformal naturality in Lemma 5 can also be deduced from Epstein's construction.) An extension operator of this form was also proposed by Ahlfors in [2], though not in this much generality and without reference to a reflection in the image. We consider Ahlfors's criterion in Section 7.

The topological fact that E_f is a global homeomorphism once it is a local homeomorphism was used on several occasions by Ahlfors. In his work, the fact that E_f is a local homeomorphism depends on showing that the Jacobian is positive, and this follows from knowing that the Beltrami coefficient in (5.8) is $\leq t < 1$. This reasoning cannot be applied in the limiting case $t = 1$.

The ideal situation would be that E_f is a homeomorphic extension of f if and only if $f(\mathbf{D})$ is a Jordan domain, but this is not the case. It is true when the metric g is real analytic, but false for C^∞ metrics.

[1] In [9] we wrote E_f for the reflection and F for the extension. We apologize for the inconsistent notation.

Theorem 6 *Let $g = e^{2\sigma}|dz|^2$ be a complete metric on $\mathbf{D}$ satisfying (ULP) and (BPJ). Suppose $||\mathcal{S}_g f||_g \leq \frac{1}{2}|K(g)|$.*
(i) If g is real analytic then $f(\mathbf{D}) = \Omega$ is a Jordan domain if and only if E_f is a homeomorphism.
(ii) If g is C^∞ then there are conformal mappings of $\mathbf{D}$ onto Jordan domains for which E_f is not a homeomorphism.

Proof. The necessity in (i) is clear. We shall prove that if E_f is not a homeomorphism then $f(\mathbf{D})$ is not a Jordan domain, and for this we appeal to the equivalent condition (c) in Theorem 5. That is, assume for some Möbius transformation M, that $u = u_{M\circ f}$ has at least two critical points in $\mathbf{D}$, say at z_1, z_2. Without loss of generality we may suppose this happens for f itself. By convexity, u then attains its absolute minimum u_0 at z_1 and z_2 and also along the geodesic segment joining them. Since the quantities are real analytic, $u = u_0$ along the entire geodesic γ through z_1 and z_2. This is the situation in Lemma 4, and it follows that $f(\zeta_1) = f(\zeta_2) = \infty$, where ζ_1 and ζ_2 are the asymptotic endpoints of γ on $\partial\mathbf{D}$. We wish to show that $\zeta_1 \neq \zeta_2$, hence Ω is not a Jordan domain. (Hence f is an extremal function and γ is an extremal geodesic.)

Suppose to the contrary that that $\zeta_1 = \zeta_2 = \zeta$. Take any point $z_0 \in \gamma$, and let γ_1 be the geodesic through z_0 normal to γ at z_0. Then γ_1 followed in one direction must end at the same asymptotic boundary point ζ, because geodesics cannot cross more than once. Let $U(t) = U(\gamma_1(t))$ with $U(0) = z_0$ and $\lim_{t\to\infty} U(t) = \zeta$. Then $U(t)$ is convex and $U'(0) = 0$. If $U(t)$ is not identically equal to u_0 for $t \geq 0$, then $U(t)$ is bounded below by some non-constant affine function, and this implies that $|f(\zeta)| < \infty$, contrary to the above. Hence $U(t) \equiv u_0$ for $t \geq 0$. Since $z_0 \in \gamma$ was arbitrary we conclude that $u = u_0$ on the component of $\mathbf{D}\backslash\gamma$ containing γ_1, and so $u = u_0$ in $\mathbf{D}$ by analyticity. Now

$$|f'| = u^{-2}e^\sigma = u_0^{-2}e^\sigma,$$

and hence the metric $g_1 = \rho^2|dw|^2$ on Ω is a constant multiple of the euclidean metric, since $\rho \circ f = u_f^2 = u_0^2$. But now $f\colon (\mathbf{D}, g) \to (\Omega, g_1)$ is an isometry, and thus g_1 is complete. This can only happen if $\Omega = \mathbf{C}$, an absurdity. This contradiction proves that $\zeta_1 \neq \zeta_2$, and hence that Ω is not a Jordan domain. We conclude that u has at most one critical point, proving the first part of the theorem.

For the proof of (ii) we construct an example of a function f satisfying (1.7) such that E_f fails to be a homeomorphism despite $f(\mathbf{D})$ being a Jordan domain. In fact, by choosing the metric g on $\mathbf{D}$ properly we can accomplish this with $f(z) = z$. Write $g = e^{2\sigma}|dz|^2$ as usual. Because $\varphi = \log|f'| = 0$ for $f(z) = z$, the inequality $||\mathcal{S}_g f||_g \leq (1/2)|K(g)|$ appears as

$$|\sigma_{zz} - \sigma_z^2| \leq \sigma_{z\bar{z}}, \tag{5.9}$$

in terms of σ alone. We want to choose σ satisfying this condition in such a way that u_f has more than one critical point. According to Theorem 5, E_f cannot then be a homeomorphism.

Let ν be defined by the equation

$$\sigma = \log\frac{1}{1-\nu}.$$

Then (5.9) is easily shown to be equivalent to

$$|\nu_{zz}| \leq \nu_{z\bar{z}} + \frac{|\nu_z|^2}{1-\nu}. \tag{5.10}$$

This inequality is in turn implied by

$$|\nu_{zz}| \leq \nu_{z\bar{z}},$$

which is itself equivalent to the euclidean convexity of ν. In summary, in order for (5.9) to hold it suffices to take $\sigma = -\log(1-\nu)$ for any convex function ν which is less than 1 in the disk.

We now take $\nu\colon \mathbf{D} \to [0,1]$ to be radially symmetric C^∞ function where, regarding ν as a function from $[0,1]$ to itself, $\nu = 0$ on a small interval $[0,\epsilon]$, $0 < \epsilon < 1$, $\nu'' \geq 0$ on $[0,1]$, $\nu(1) = 1$ and $\nu'(1) < \infty$. Because of this last condition

$$\int_0^1 \frac{dr}{1-\nu(r)} = \infty,$$

making the corresponding metric $e^{2\sigma}|dz|^2$ complete. The resulting function u_f will have all z with $|z| < \epsilon$ as critical points. This completes the construction, and with it the proof of the theorem.

Finally, observe what happens with the reflection Λ_f when there is an extremal function f and an extremal geodesic γ. Suppose that γ has endpoints $\zeta_1, \zeta_2 \in \partial\mathbf{D}$, and normalize f so that $f(\zeta_1) = f(\zeta_2) = \infty$. Let γ^* be the reflection of γ in $|z| = 1$. Corollary 4 states that $\operatorname{grad} u_f$ vanishes along γ, and so from the definition of Λ_f, and the relation $\rho_f \circ f = u_f^2$, we see that that Λ_f is identically ∞ along γ^*. That is, Λ_f collapses the reflection of an extremal geodesic to a point. By conformal naturality, (5.3), this holds regardless of whether f is normalized.

6 Visible Boundary

We recall the unique limit point property (ULP), (ULP*) and the boundary point joining property (BPJ) from Section 1. In this section we give several sufficient conditions in terms of the conformal factor σ for the metric $g = e^{2\sigma}|dz|^2$ to have these properties. We continue to assume that the curvature is ≤ 0. We let r and θ denote the usual polar coordinates on $\mathbf{D}$.

Theorem 7 *(i) Suppose that $g = e^{2\sigma} g_0$ satisfies*

$$\begin{aligned} \sigma_r &> 0 \\ |\sigma_\theta| &\leq C(1-r)^{-\alpha}\sigma_r, \end{aligned}$$

for some constants $C \geq 0$ and $\alpha \in [0,1)$. Then g satisfies ULP.
(ii) Suppose that $\sigma_r(z) \to \infty$ as $|z| \to 1$ and for some annulus $r_0 \leq |z| < 1$ and a constant $M < \infty$ that $|\sigma_\theta(re^{i\theta})| \leq M$. Then (ULP) and (BPJ) hold.*

Proof. (i). We observe first that that a euclidean disk $|z| \leq r$ is convex in the metric g, for the condtion $\sigma_r > 0$ implies that $|z| = r$ has positve geodesic curvature. As a consequence, $|z|$ cannot have a local maximum in $\mathbf{D}$ along a maximally extended geodesic. Thus, if properly traced, $|z|$ must be increasing along a tail of the geodesic.

In order to prove (ULP) we shall use polar coordinates to analyze the geodesic equation. With the metric in the form $e^{2\sigma}(dr^2 + r^2 d\theta^2)$ the Christoffel symbols are

$$\begin{array}{lll} \Gamma^r_{rr} = \sigma_r, & \Gamma^r_{r\theta} = \sigma_\theta, & \Gamma^r_{\theta\theta} = -r - r^2\sigma_r, \\ \Gamma^\theta_{rr} = -\frac{1}{r^2}\sigma_\theta, & \Gamma^\theta_{r\theta} = \frac{1}{r} + \sigma_r, & \Gamma^\theta_{\theta\theta} = \sigma_\theta. \end{array}$$

Let $\gamma(t) = (r(t), \theta(t))$ be a unit speed geodesic. Then $\dot r^2 + r^2\dot\theta^2 = e^{-2\sigma}$ and the geodesic equations become

$$\begin{aligned} \ddot r + \sigma_r \dot r^2 + 2\sigma_\theta \dot r\dot\theta - (r + r^2\sigma_r)\dot\theta^2 &= 0, \\ \ddot\theta - \frac{1}{r^2}\sigma_\theta \dot r^2 + 2(\frac{1}{r} + \sigma_r)\dot r\dot\theta + \sigma_\theta\dot\theta^2 &= 0. \end{aligned}$$

We write this as the following first order system in the variables r, θ, ξ, and η:

$$\begin{aligned}\dot{r} &= e^{-2\sigma}\xi, \\ \dot{\theta} &= \frac{1}{r^2}e^{-2\sigma}\eta, \\ \dot{\xi} &= \sigma_r + e^{2\sigma}r\dot{\theta}^2, \\ \dot{\eta} &= \sigma_\theta.\end{aligned}$$

Fix a base point $z_0 \neq 0$. Let γ be a geodesic starting at z_0, and suppose the initial conditions are $r(0) = r_0 = |z_0| > 0$, $\theta(0) = \theta_0$, $\xi(0) = \xi_0$ and $\eta(0) = \eta_0$, with $\xi_0 > 0$, *i.e.*, the geodesic is initially moving toward the boundary. Since $\dot{\xi} > 0$ we have $\xi(t) > 0$ for all t, hence $r(t)$ is strictly increasing. It is therefore possible to consider θ as a function of r along the curve.

We want to estimate $|d\theta/dr|$. Since $|d\theta/dr| = |\dot{\theta}/\dot{r}| = |\eta/r^2\xi|$, we need to bound $|\eta/\xi|$. For this,

$$|\dot{\eta}| = |\sigma_\theta| \leq C(1-r)^{-\alpha}\sigma_r \leq C(1-r)^{-\alpha}\dot{\xi},$$

hence

$$\begin{aligned}|\eta(t) - \eta_0| &\leq C\int_0^t (1-r(s))^{-\alpha}\dot{\xi}(s)\,ds \\ &\leq C(1-r(t))^{-\alpha}\int_0^t \dot{\xi}(s)\,ds,\end{aligned}$$

because $r(s)$ is increasing. Thus

$$|\eta(t) - \eta_0| \leq C(1-r(t))^{-\alpha}(\xi(t) - \xi_0) \leq C(1-r(t))^{-\alpha}\xi(t),$$

or

$$|\eta(t)| \leq C(1-r(t))^{-\alpha}\xi(t) + |\eta_0|,$$

so

$$\left|\frac{\eta(t)}{\xi(t)}\right| \leq C(1-r(t))^{-\alpha} + \frac{|\eta_0|}{\xi(t)}.$$

Since

$$\frac{|\eta_0|}{\xi(t)} \leq \frac{|\eta_0|}{\xi_0},$$

it follows, for some constant C_1 depending on r_0 and $|\eta_0|/\xi_0$, that

$$\left|\frac{\eta(t)}{\xi(t)}\right| \leq C_1(1-r(t))^{-\alpha}.$$

With this,

$$\left|\frac{d\theta}{dr}\right| = \left|\frac{\eta}{r^2\xi}\right| \leq \frac{C_1}{r^2}(1-r)^{-\alpha} \leq C_2(1-r)^{-\alpha}.$$

Now,

$$|\theta(r) - \theta(r_0)| \leq \int_{r_0}^r \left|\frac{d\theta}{dr}\right|\,dr,$$

and by the estimate above the integral converges as $r \to 1$. Therefore $\theta(r)$ has a limit as $r \to 1$, and γ has a unique limit point on $\partial\mathbf{D}$.

We next want to show that the end point of γ on $\partial\mathbf{D}$ varies continuously with the initial direction. The estimate we need for this is, essentially, a bound on the euclidean diameter of the tail of a geodesic. Take $0 \le t_1 < t_2$, with $r_0 \le r_1 = |\gamma(t_1)| < |\gamma(t_2)| = r_2$. Then

$$\begin{aligned} |\gamma(t_2) - \gamma(t_1)| &\le \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr \\ &\le \int_{r_1}^{r_2} \sqrt{1 + C_2^2 r^2 (1-r)^{-2\alpha}}\, dr \\ &\le C_3 \int_{r_1}^{1} (1-r)^{-\alpha}\, dr \le C_0 (1-r_1)^{1-\alpha}. \end{aligned}$$

The constant C_0 depends on α and on the initial data at the base point z_0. In particular, the euclidean diameter of the tail of γ tends to zero.

Let γ_0 be a geodesic starting at z_0 with initial direction θ_0 and ending at a point ζ_0 on $\partial\mathbf{D}$. We show continuity at θ_0.

Let $\epsilon > 0$. Let r_1 be such that

$$C_0(1-r_1)^{1-\alpha} < \frac{\epsilon}{3},$$

with C_0 as in the estimate above. Choose r_2 so that $0 < r_2 - r_1 < \epsilon/3$. There exists a $t_1 \ge 0$ such that

$$|\zeta_0 - \gamma_0(t)| < \frac{\epsilon}{3}, \qquad \text{and} \qquad |\gamma_0(t)| \ge r_1,$$

if $t \ge t_1$.

Let $\gamma(t)$ be another geodesic starting at z_0 with initial direction θ. By continuity of the solution of the geodesic equation in the parameters, there exists a $\delta > 0$ such that $|\theta - \theta_0| < \delta$ implies for $0 \le t \le t_1$ that,

$$|\gamma(t) - \gamma_0(t)| < r_2 - r_1 < \frac{\epsilon}{3}.$$

Hence

$$|\gamma(t_1)| > |\gamma_0(t_1)| - (r_2 - r_1) \ge r_1.$$

If δ is small enough the constant, say C_0', entering into the estimate on the tail of γ, will also satisfy $C_0'(1-r_1)^{1-\alpha} < \epsilon/3$, and thus γ has a small tail after $\gamma(t_1)$. So, for $t \ge t_1$,

$$|\zeta_0 - \gamma(t)| \le |\zeta_0 - \gamma_0(t_1)| + |\gamma_0(t_1) - \gamma(t_1)| + |\gamma(t_1) - \gamma(t)| < \epsilon.$$

Thus the endpoint $\zeta \in \partial\mathbf{D}$ of γ has $|\zeta - \zeta_0| < \epsilon$.

Finally, the estimate on the tails also implies that every point on $\partial\mathbf{D}$ is the limit point of a geodesic. Let ζ_0 be a point on $\partial\mathbf{D}$ that can be reached by a geodesic from z_0. We show that we can reach some other point on $\partial\mathbf{D}$, and this implies that all points on the boundary are visible. Let $z_1 \in \mathbf{D}$ be very close to, say, $-\zeta_0$. Choose a disk $|z| \le r$ containing both z_0 and z_1. Since this disk is convex in the metric g, the geodesic from z_0 to z_1 stays in the disk. If $|z_1|$ is sufficiently close to 1 then the tail of γ from z_1 to the boundary will be small, making it impossible for its endpoint to be ζ_0. This completes the proof of part (i) of the theorem.

(ii). We first prove (ULP*), for which we need only consider the case when the diameter of the disk is finite. The euclidean curvature of a geodesic is given by the normal derivative $\partial\sigma/\partial n$. Since $\sigma_r \to \infty$ while σ_θ is bounded, it follows that the tangent vector to a geodesic tends to the radial direction as the geodesic tends to the boundary. Then it is easy to see that the ratio of the euclidean lengths of tails of geodesics with close initial conditions is uniformly bounded. Now, because $|\sigma_\theta| \le M$, a euclidean rotation is a quasi-isometry for the metric g near the boundary. Thus the lengths of such tails must tend uniformly to zero, and one deduces the continuity of the length function in (ULP*) directly.

To prove that any two points on $\partial\mathbf{D}$ can be joined by a geodesic in $\mathbf{D}$, whether or not g is complete, we first establish a property of the total curvature near the boundary. Namely:
If U is a relatively open subset of $\overline{\mathbf{D}}$ containing a polar rectangle $R = \{re^{i\theta} : r_1 \le r \le 1,\, \alpha \le \theta \le \beta\}$, then

$$\int_U K(g)\, dA_g = -\infty. \tag{6.1}$$

For this, let $R' = \{re^{i\theta} : r_1 \le r \le r_2 < 1,\, \alpha \le \theta \le \beta\} \subset R$. Then

$$\int_{R'} K(g)\, dA_g = -\iint_{R'} \Delta\sigma\, dx\, dy = -\int_{\partial R'} \frac{\partial \sigma}{\partial n}\, |dz|.$$

For the line integral, the contributions along the radial sides of R' are uniformly bounded by virtue of the assumption $|\sigma_\theta(re^{i\theta})| \le M$. The arc of the inner circle is fixed, while along the outer arc, $\partial\sigma/\partial n = \sigma_r \to \infty$ as $r_2 \to 1$. We conclude that

$$\int_R K(g)\, dA_g \to -\infty \quad \text{as} \quad r_2 \to 1$$

hence

$$\int_U K(g)\, dA_g = -\infty,$$

which proves (6.1).

We sketch how the unique limit point property and (6.1) come into play in proving that any two points on $\partial\mathbf{D}$ can be joined by a geodesic. This is a standard part of the more general theory of visibility manifolds, and it is included here for the convenience of the reader. Let $z_0 \in \mathbf{D}$ be fixed. We consider all geodesics starting at z_0. By (ULP), any such geodesic $\gamma(t)$, $0 \le t \le T_\gamma$, determines a point $w = \lim_{t \to T_\gamma} \gamma(t) \in \partial\mathbf{D}$. We may consider w as a function of $\gamma'(0)$ in the tangent space $T_{z_0}\mathbf{D} \cong S^1 = \partial\mathbf{D}$. As in the proof of Theorem 3, w depends monotonically and continuously on $\gamma'_{z_0}(0)$ and all such limit points must cover $\partial\mathbf{D}$.

Let $w_1, w_2 \in \partial\mathbf{D}$, and let γ_1 and γ_2 be two geodesics starting at z_0 which have w_1 and w_2 as asymptotic limits, respectively. Let $a_n = \gamma_1(t_n)$, $b_n = \gamma_2(t_n{}')$, $a_n \to w_1$, $b_n \to w_2$, and let $\Gamma_n = \Gamma_n(t)$ be the (unique) geodesic joining a_n to b_n. (Since $\sigma_r \to \infty$ it follows easily that such geodesics exist and lie in $\mathbf{D}$.) A direct application of the Gauss-Bonnet theorem gives that the integrals

$$\int_{T_n} K(g)\, dA_g$$

are uniformly bounded below, where T_n is the triangle bounded by $\gamma_1|_{[0,t_n]}$, $\gamma_2|_{[0,t_n{}']}$, and Γ_n. By (6.1) Γ_n cannot converge to $\partial\mathbf{D}$. Hence, by passing to a subsequence of the Γ_n, there is sequence $\{z_n\}$ on Γ_n with $z_n \to z_1 \in \mathbf{D}$, and also with the tangent vectors Γ_n' at z_n converging to a direction θ_1 at z_1. Then the geodesic through z_1 with direction θ_1 is the desired geodesic; when followed forward and backward from z_1 it will have w_1 and w_2 as asymptotic limits. This completes the proof of Theorem 7.

Next, we show that for complete metrics a condition on just the angular derivative of the conformal factor is sufficient to guarantee (ULP) and (BPJ). In many examples the conformal factor is a radial function, so this result is particularly useful.

Theorem 8 *Let $g = e^{2\sigma} g_0$ be complete. Suppose for some annulus $0 < r_0 \leq |z| < 1$ and for some constant $C < \infty$ that*

$$\sigma_{\theta\theta}(re^{i\theta}) \leq C. \tag{6.2}$$

Then (ULP) and (BPJ) hold.

Proof. Let $r_0 \leq r < 1$. On every circle $|z| = r$ there is at least one point where σ_θ vanishes, and hence $\sigma_\theta \leq 2\pi C$. But then also $\sigma_\theta \geq -2\pi C$, and thus

$$|\sigma_\theta(re^{i\theta})| \leq 2\pi C. \tag{6.3}$$

From the curvature condition $-e^{-2\sigma}\Delta\sigma \leq 0$ we then have that

$$\sigma_{rr} + \frac{1}{r}\sigma_r + \frac{1}{r^2}\sigma_{\theta\theta} \geq 0, \quad \text{or} \quad r\sigma_{rr} + \sigma_r \geq -\frac{C}{r} \geq -\frac{C}{r_0}.$$

We write this as

$$(r\sigma_r)_r \geq -\frac{C}{r_0}, \tag{6.4}$$

from which

$$r\sigma_r \geq -c > -\infty \quad \text{on} \quad r_0 \leq |z| < 1. \tag{6.5}$$

Next, as g is complete,

$$\int_0^1 e^{\sigma(x)}\, dx = \infty,$$

and hence there exists a sequence $\{x_n\}$, $x_n \to 1$, with $\sigma(x_n) \to \infty$. We may assume that x_n is increasing and that $\sigma(x_{n+1}) - \sigma(x_n) \geq n$. It follows from (6.3) that on any radius $\sigma(x_n e^{i\theta}) \to \infty$, and that $\sigma(x_{n+1}e^{i\theta}) - \sigma(x_n e^{i\theta}) \geq n - 4\pi C$ if $x_n \geq r_0$. For θ fixed, the mean value theorem then yields a sequence $\{y_n(\theta)e^{i\theta}\}$, $x_n \leq y_n(\theta) \leq x_{n+1}$, such that $\sigma_r(y_n(\theta)e^{i\theta}) \to \infty$. Now from (6.4) we deduce that

$$\sigma_r(z) \to \infty \quad \text{as} \quad |z| \to 1. \tag{6.6}$$

The preceding theorem now applies.

The conditions in Theorems 7 and 8 are certainly not optimal, but they are well suited to many applications and examples. Separately or together they seem to express the fact that the metrics we need to work with are 'asymptotically radial', but we do not put forward a definition.

7 An Example

We consider the family of metrics

$$g = \frac{|dz|^2}{(1-|z|^2)^{2t}}, \quad 0 < t < 1,$$

of negative curvature, for which $\mathbf{D}$ has finite diameter

$$\delta = 2\int_0^1 \frac{dx}{(1-x^2)^t} = \sqrt{\pi}\frac{\Gamma(1-t)}{\Gamma(\frac{3}{2}-t)}.$$

The corresponding univalence criterion reads

$$\left|Sf(z) - \frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2}\right| \le \frac{2t}{(1-|z|^2)^2} + \frac{2\pi^2}{\delta^2}\frac{1}{(1-|z|^2)^{2t}}. \tag{7.1}$$

See [2] and [10].

The function $\sigma(z) = -t\log(1-|z|^2)$ is radial, and

$$\sigma_r = \frac{2tr}{1-r^2} \to \infty, \quad r \to 1. \tag{7.2}$$

It follows from Theorem 7 that g satisfies both the properties (ULP*) and (BPJ).

Let

$$F(z) = \frac{1}{c}\tan\left\{c\int_0^z \frac{d\zeta}{(1-\zeta^2)^t}\right\}, \quad c = \frac{\pi}{\delta}. \tag{7.3}$$

It was shown in [6] that this function, which satisfies (7.1) with equality along $(-1,1)$, satisfies the inequality in the full disk if and only if $1/2 \le t < 1$. Thus for this range of t, F is an extremal function for (7.1) and $(-1,1)$ is an extremal geodesic. By rotating F we get extremal functions and extremal geodesics for any diameter. Furthermore, one can check that the only geodesics having length δ are precisely the euclidean diameters. It then follows from Lemma 2 that F and its rotations account for *all* the extremal functions for the criterion (7.1) for $0 < t < 1$. In particular, for $0 < t < 1/2$ there are *no* extremal functions and any f satisfying (7.1) for a t in this range will map the disk onto a Jordan domain.

Now let f satisfy (7.1) and suppose that f is not an extremal function. We normalize so that $u_f(0) = 1$ and $\operatorname{grad} u_f(0) = 0$. (Since $e^{\sigma(0)} = 1$ and 0 is a critical point for σ, this normalization for f is equivalent to $|f'(0)| = 1$, $f''(0) = 0$.) Let $\gamma = \gamma(t)$ be any radial segment, with $\gamma(0) = 0$. Then for $U(t) = u_f(\gamma(t))$ we have

$$U'' \ge -\frac{\pi^2}{\delta^2}U,$$

and since f is not extremal it follows that

$$\inf_{0\le t<\delta/2} U(t) > 0.$$

It is not difficult to show that this infimum is uniformly bounded below, independent of γ. Hence

$$\inf_{z\in\mathbf{D}} u_f(z) = a > 0,$$

and therefore

$$|f'(z)| \le \frac{1}{a^2} e^{\sigma(z)} = \frac{1}{a^2} \frac{1}{(1-|z|^2)^t}.$$

This inequality implies that $f(\mathbf{D})$ is a bounded (Jordan) domain, and that f admits a $(1-t)$-Hölder continuous extension to $\overline{\mathbf{D}}$; see *e.g.* [8].

Similar remarks apply to the family of *complete* metrics

$$g = \frac{|dz|^2}{(1-|z|^2)^{2t}},$$

where this time $1 < t \le 2$. For the diameter $(-1, 1)$ an extremal is again given by

$$F(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^t},$$

but now F satisfies (1.5), which translates to

$$\left| Sf - \frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2} \right| \le \frac{2t}{(1-|z|^2)^2}, \tag{7.4}$$

in $\mathbf{D}$ for the full range $1 < t \le 2$; see [6], and [2]. In this case we do not know if there are any other extremal functions.

Suppose f is a non-extremal satisfying (7.4) and normalized by $f''(0) = 0$. Then the convexity of u_f gives

$$|f'(z)| \le \frac{e^{\sigma(z)}}{(a + b d_g(0,z))^2},$$

for some constants a, b. One checks that

$$d_g(0,z) \sim \frac{1}{(1-|z|)^{t-1}}, \quad |z| \to 1,$$

and using this we get,

$$|f'(z)| = O\left(\frac{1}{(1-|z|)^{2-t}} \right).$$

This implies that $f(\mathbf{D})$ is bounded and has a $(t-1)$-Hölder continuous extension to $\overline{\mathbf{D}}$. A homeomorphic extension for f is induced by the reflection

$$\Lambda_f(f(z)) = f(z) + \frac{(1-|z|^2) f'(z)}{t\bar{z} - (1-|z|^2)\frac{1}{2}\frac{f''}{f'}(z)}.$$

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